Decoherence Subjective Round 2023

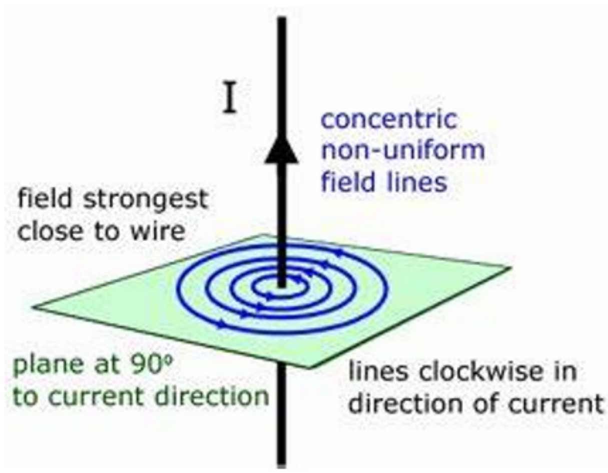
Electromagnetism

1. Magnetic field lines were introduced by Michael Faraday who called them lines of force. They are elegant graphic depictions of Magnetic field due to a source. Finding equation for Magnetic field lines is an interesting task. The basic principle is that the magnetic field line curve is always tangent to Magnetic field. Mathematically, it can be expressed as

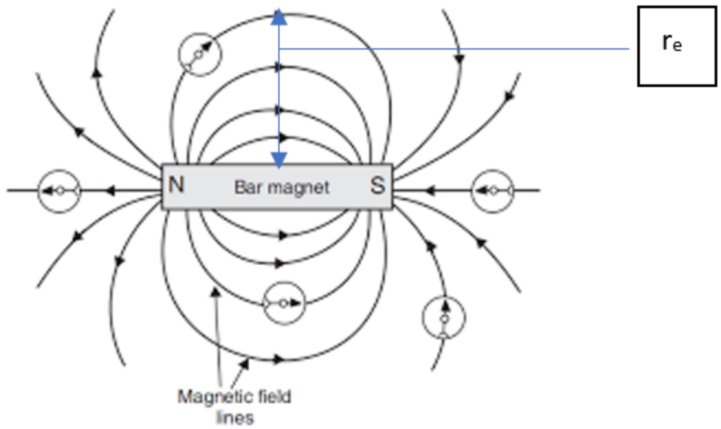
$$\mathbf{F}(\mathbf{r}(t)) = \frac{d\mathbf{r}(t)}{dt}$$

where $\mathbf{F}$ is the field vector and $\mathbf{r}$ is the position vector.

(a) The given image indicates field lines due to infinitely long wire. They are well known to be concentric circles. Prove that magnetic field lines due to infinitely long current carrying wire are concentric circles. You might wish to exploit the z -axis invariant property of curves in this case.



(b) The given image indicates magnetic field lines due to a bar magnetic or magnetic dipole. Note that field lines are perpendicular to the equatorial plane. Prove that the equation for field lines is given by $r = r_e sin^2(\theta)$, where r is the distance from centre; θ is polar angle assuming dipole along z axis and r_e is perpendicular distance of field line from dipole at equator(indicated in diagram). One must remember that sometimes a geometric argument using a
diagram can simplify our task



(c) Van Allen radiation belt is a zone of energetic charged particles around Earth, named after its discoverer James Van Allen. Using the above question now we wish to investigate the movement of charged particles in Van Allen radiation belt of a newly discovered planet. We study Gyro motion in which a particle gyrates around a magnetic field line i.e. moves in an almost helical pattern such that the axis of curve is along magnetic field line (The axis will be curved and at any instant motion can be considered helical with axis along the tangent). Model the planet as an ideal dipole with dipole moment due to this planet as $4.39 \times 10^{22} Am^2$. The radius of planet is 4983 km. Consider a general magnetic field line and find the frequency and radius of helical motion as the particle gyrates along the magnetic field line. Assume the mass and charge of the particle as m and q. Assume that speed along the axial direction is v_1 and speed perpendicular to axis is v_2. Define a variable 'pitch angle' denoted by p_θ which is the angle made by net velocity vector with with plane perpendicular to axis . Note that $p_\theta = tan^{-1}(\frac{v_2}{v_1})$ Consider the perpendicular distance of the field line at the equator to be r_e. Neglect the rotational effects of Earth.

(d) Now consider the particle to be a proton. It has been observed that the magnetic moment that the proton generates due to its motion perpendicular to axis remains constant. Consider v_1 and v_2 at equator to be v_{1eq} and v_{2eq}. After reaching a specific latitude the proton's axial speed becomes θ. Qualitatively explain what might be the reason for this. Then determine this latitude. Neglect gravitational interactions of proton with planet

(e) Determine the area of the planet from which the particle would be 'visible' at the point it turns back ($v_1 = 0$) ?(You can neglect effect of Earth's atmosphere on light rays and assume that particle is visible at those locations where rays after getting reflected from particle reach. Assume dimensions of particle are sufficiently larger than wavelength of light)

(f) Determine the initial conditions at Equator for which this 'point of return ' exists (i.e. the proton must not bump into the planet before it reaches such a point)

2. The method of images is closely related to the uniqueness theorem in electrostatics. One can also apply a similar artifice in magnetostatics. First, we shall prove the uniqueness theorem in magnetostatics.

   (a) Suppose a surface S encloses a volume, and tangential and normal B is given on every surface part. Prove that the magnetic field inside volume V is uniquely determined.

   (b) As we'll see, a superconductor differs from a perfect conductor. Suppose we have a perfect conductor that has **E**=0 inside. Prove that the magnetic field must be constant inside a perfect conductor

   (c) For a superconductor,**B** is not only constant but this constant is 0. Prove that the superconductor cannot have any current inside.

   (d) Suppose a magnetic dipole **P** is fixed at height h in gravity-free space, above a perfectly superconducting material that occupies space for z¡=0. Find force due to magnet on a superconductor.

   (e) Now, suppose that the dipole is free to move along the z-axis, but the orientation of the dipole is fixed (parallel to the superconducting surface). Find the height at which the dipole is in equilibrium.

   (f) If the dipole is displaced slightly from its equilibrium position. With what frequency does the dipole oscillate?

Astronomical Adventure

3. You are attending an astronomy camp with your friends, and there you meet lot of curious brains just like yours. Since you cleared the Decoherence 1st round, you have to answer some interesting questions. So try to follow the story given below and answern the questions.

Day 1

Star Party

Q 3.1

On some day, you are looking up at night sky. You see an interesting phenomenon called "Moon Occultation". You see the star very near to the Moon, what will be the longest possible duration (in minutes) of the occultation?

Q 3.2

After the Moon occultation you see a bright star. Using the stellarium application, you look for it's Right Ascension (Ra) and Declination (Dec): they were (15h 57m08.5s, +48°29'22.7"). Irrespective from your location, will this source be visible from Bangalore? If yes, then what is the best time of the year to observe it from Bangalore using an optical telescope?

Q 3.3

Now your friend, who's with you doing stargazing is wondering if we can find the location of star in form of galactic latitude and longitude, and actually we can, derive the set of equations which will converted (Ra, Dec) $\rightarrow$ (Galactic latitude, Galactic longitude) and find same for the star whose coordinates are given above.

Day 2

Fun with Sun

Q 3.4

Do you know that path on which sun moves on the celestical equator is called Ecliptic (You can search about this), now using this piece of information, how can you find directions if you get lost somewhere. (Hint: Shadows)

Q 3.5

Now the very interesting thing about today is, it is the day of equinox so your task is to find the latitude of your place and find the radius of Earth.

Q 3.6

So, during the camp you made several friends and everyone is from different parts of India, you people are discussing about the shadows and someone asks the following questions

1. I observed sunrise at 05:00 on 13th June.

2. I observed sunset at 17:00 on 24th December.

3. I observed sunset at 17:47 on 24th December, which was the second earliest out of the five cities on that day.

4. I observed sunset at 19:00 on 1st September, which was the last sunset on that day.

5. At local noon on 21st June, shadow at my location was the longest amongst all.

6. The shortest shadow of the year at my location was observed on 21st June.

7. On 1st July, I had a longer day as compared to other observers.

8. On 1st February, I had a longer day as compared to other observers.

Given Information

Observer	Coordinates3
Akshank	22.57 °N, 88.36 °E
Gaurav	23.7337 °N, 69.8597 °E
Hemansh	33.2778 °N, 75.3412 °E
Kenil	9.93 °N, 76.27 °E
Sahil	19.08 °N, 72.88 °E

Now, you have to solve this riddle by giving answers to these 8 question with perfect reasoning.

Q 3.7

During the sunset you observe Venus, Jupiter and Saturn, and you see a phase of Venus somewhat similar to that of moon, and someone asked that "Why do we see phase for Venus but not for Jupiter or Saturn ?"

Day 3

Magic of Moon

Q 3.8

And now comes the last night of camp, this time you are doing stargazing at seashore to escape the city lights. Moon is setting, and you see lot stars shining in the night sky and this time, the friend you made Hemansh is asking "If the mass of the moon is doubled and the length of the month is halved, leaving all Earth characteristics and Fundamental constants same, How will the height of tides change ?"

Q 3.9

During the stargazing, Sahil arrives, he looks at the moon and asks that "Can we find out in which day the moon is in its lunar cycle (Lunar month)?", find a way to ans his question. (your answer should cover each and every aspect of his question, so think and answer)

Malcolm's Mortician

4. Malcolm Marshall, the bane of batsmen for years had always been looked up as a terror with a certain projectile. And the source of that terror? An interesting contraption we call hand.

   Naturally there was a world of speculation about his famed arm works. The mortician, when left alone with Marshall's earthly remains, decided to unravel the mystery. Fortunately this particular mortician remembered his Lagrangian lessons from childhood and he set out on his project.

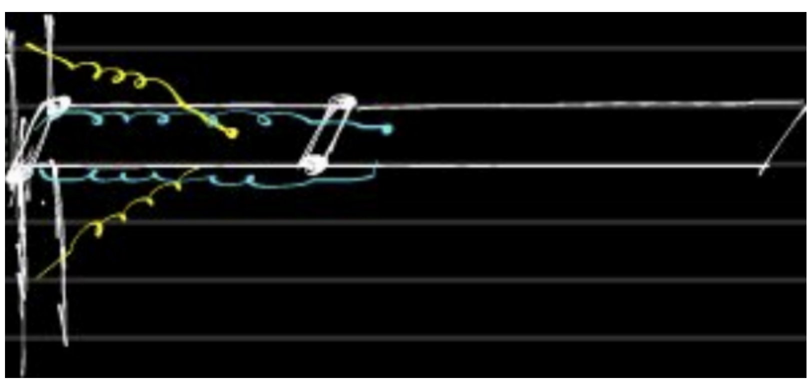
Part A

Why don't you embark on a side quest too? Try and create a working model of the hand.

You have the following equipment to work with:

- 2 wooden planks
- 3 springs: each one has the same spring constant
- As many bolts as you require
- A stable support
- 2 hinges: one fixed on the support, another with 2 ends free.

The model should be reasonably functional in a way similar to the biological functioning of muscles, and each plank should have bidirectional movement.



Part B

Now that you have the model, let's look into the actual motive of the model- to estimate how Marshall threw that ball so hard.
The support pivot has a range with amplitude 30° in either direction while the other pivot moves 150° in one direction.

Where does the kinetic energy of the projectile come from?

Now that you know the working model and have an idea about the mechanism let's move on to the math. Work out, in terms of k and other constants, the maximum energy you can impart to a projectile using this model.

A necessary assumption for this calculation is the existence of constant force or constant torque. Which assumption is correct and why?

Note: Take care to use a rudimentary idea of how the arm works- maybe even move your own hand and feel it!

Part C

Now that we have a net energy we have to account for how the batsman felt too! So, find out the maximum range of the projectile considering viscous drag from air $F_v = -\beta v$,

where β is a constant $= 6\pi\eta r$, η is the viscosity constant of air, and r is the radius of the ball.
Note: You can ignore buoyant effect of air- given the high density of cricket balls.

Part D

The mortician know rests in a chair, finally at peace. But his reverie didn't last long- as immediately after, his friends sent him a video of a Wasim Akram reverse swing- which had shaken the entire world. The passion of physics still hot in his vein our mortician sets back to work.

What is the science behind reverse swing?

Considering a 22 yard pitch, say Wasim bowls an yorker. Given the initial velocity $\mathbf{v}$ and that the ball was released horizontally what will be the trajectory of the ball?

Note:

Take the following effects into consideration: Viscous drag, gravity and differential pressure forces.

Strange Gas X

5. Suppose we have a gas X which is found to satisfy the following relations, $u = AT^4$, u is internal energy per unit volume of the gas, pressure $P = u/3$.

PART A

(a) X is enclosed within a spherical shell of radius R at temperature T_0. Find the work done by the gas in reversible isothermal expansion of X from radius R to $2R$.

(b) If the shell undergoes a reversible adiabatic expansion from radius R to 2R, find the final temperature of X if the initial temperature is T_0.

(c) X is enclosed within a spherical shell of radius R. This shell is surrounded by an adiabatic spherical shell of radius $4R$. If the inner shell wall is broken, X undergoes adiabatic free expansion into the outer shell. If the initial temperature of X is T_0, what is the final temperature attained after the expansion mentioned?

PART B

(d) X is enclosed within a long fixed adiabatic cylinder C and is trapped between two adiabatic pistons A and B of masses m_1 and m_2 respectively. X occupies volume V_0 and is at temperature T_0 initially. Pistons A and B can slide inside C without friction. At the initial instant, B is given velocity v_0 while A is at rest. Find the maximum temperature of X during the motion (denote volume occupied by X when maximum temperature is reached by V_f).

Path integral

6. The path integral formulation is a description in quantum mechanics that calculates the probability amplitude to go from a given initial state to a given final state by adding the amplitude of all the possible trajectories that the system can take between the two states. It replaces the classical notion of a single, unique classical trajectory for a system with a sum (or functional integral), over an infinity of quantum-mechanically possible trajectories to compute a quantum amplitude.

   You might find it helpful to watch some videos on path integral approach to quantum mechanics on youtube if you are not already familiar to the basic idea behind such calculations. In this problem we shall be try to show how the maths of path integral actually works via an example.

   6.1: Finding the wave equation

   Let us try to calculate probability amplitude of a spin-zero free particle to to go from one point in space to another point. To keep the calculations simple let us assume the space to have only 1 space direction (i.e. 1 dimensional) and a time dimension. Take value of c to be 1 for this problem.

   Suppose the particle can travel back and forth only with speed of light. Let us consider the particle to travel only forward in time for this problem. Then in the xt plane all trajectories shuttle back and forth with slopes of ±45°, as in Fig. 1. The amplitude for such a path can be defined as follows: Suppose time is divided into small equal steps of length e. Suppose reversals of path direction can occur only at the boundaries of these steps, i.e., at $t = t_A + ne$, where n is an integer and A is the intial point.

   Let there be R such reversals that the particle take on a give path then the contribution of that path to the probability amplitude will be calculated by the following expression:

$$\phi = i\alpha^R \tag{1}$$

   See the Fig. 2 for an example of calculating ψ for some points

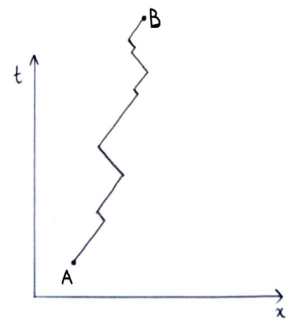


Figure 1: All lines at 45^o angle

Let's find the probability amplitude (ψ) of a particle to go from point A=(0,0) to point B=(x,t) on the xt plane , where x=Xe and t=Te

Note:

- X,$T \in \mathbb{Z}$
- x and t are a constant in this problem and in the limit e tends to zero X, T changes in a way so as to keep the value of x and t remains the same.

6.1.1

Find the condition X and T should satisfy so that the probability of finding the particle on that point is non-zero.

6.1.2

There is a rectangular region within which all the paths lies. Draw the respective region for the initial and the final points in the discrete space-time lattice below:

6.1.3

Calculate the number of paths connecting initial and final points having R reversals

Hint:

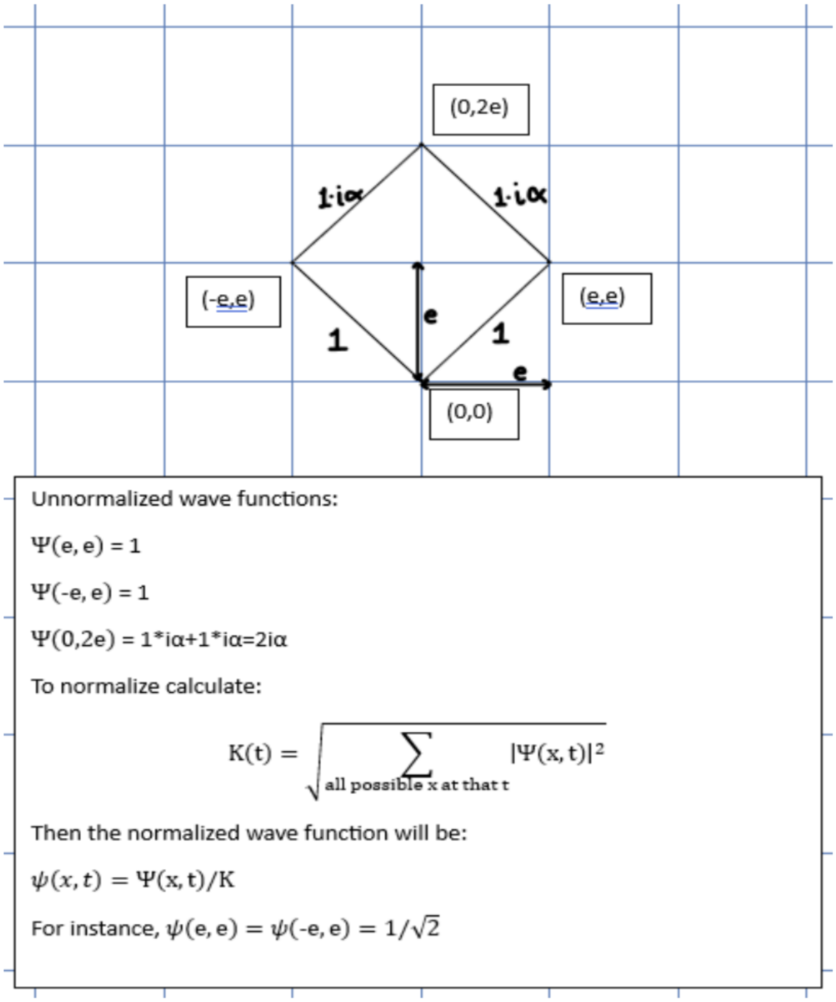
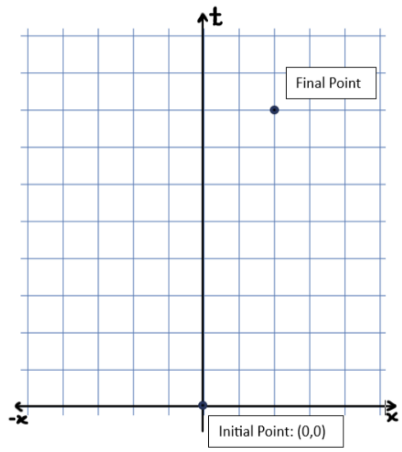


Figure 2: An example



- It becomes a simple permutation and combination question if we count the number of paths having R reversals by breaking them into four cases:
  (a) Paths that start with positive velocity and end with positive velocity ($N(R)_{++}$)
  (b) Paths that start with positive velocity but end with negative velocity ($N(R)_{+-}$)
  (c) Paths that start with negative velocity and end with negative velocity ($N(R)_{--}$)
  (d) Paths that start with negative velocity but end with positive velocity ($N(R)_{-+}$)

  We get, $N(R) = [N(R)_{++}] + [N(R)_{+-}] + [N(R)_{--}] + [N(R)_{-+}]$

6.1.4

Define,

$$a = \frac{T + |X|}{2} \tag{2}$$

$$b = \frac{T - |X|}{2} \tag{3}$$

Show that ψ can be written as the constant term in the binomial expression:

$$\left(1 + \frac{1}{z^2}\right)^{a-1} \left(1 + (i\alpha z)^2\right)^{b-1} \left(\frac{1}{z} + i\alpha z\right)^2 \tag{4}$$

Hint:

- It is best to calculate ψ by counting the number of paths $N(R)$ having R reversals connecting the initial and final point first and then writing ψ as shown a summation shown below:

$$\psi = \sum_{all\ possible\ R} N(R)(i\alpha^R) \tag{5}$$

6.1.5

Find the differential equation that ψ will satisfies at the point (x,t) in the limit e tend to zero.

Hint:

- We can write the following derivatives in a discrete way in the limit e tends to zero:
  - $\frac{\partial\psi(x,t)}{\partial x} = \frac{\psi(x+2e,t)-\psi(x,t)}{2e}$
  - $\frac{\partial\psi(x,t)}{\partial t} = \frac{\psi(x,t+2e)-\psi(x,t)}{2e}$
  - $\frac{\partial^2\psi(x,t)}{\partial x^2} = \frac{\psi(x+2e,t)+\psi(x-2e,t)-2\psi(x,t)}{(2e)^2}$
  - $\frac{\partial^2\psi(x,t)}{\partial t^2} = \frac{\psi(x,t+2e)+\psi(x,t-2e)-2\psi(x,t)}{(2e)^2}$
- Try to find an equation (L=0) in terms of the binomial expansion partial derivatives of ψ and ψ itself such that there is no constant term when we expand the binomial expressions. Then substitute back the partial derivatives of ψ and ψ for in place of the binomial expressions.
- The value of $\frac{\alpha}{e}$ must be a finite constant as e tends to zero.
- The equation must have same order in x and t since it is a relativistically correct equation.

Now we have our desired differential equation that ψ given by Eq.(5) must satisfy.

6.2: Solving it

6.2.1

Let's try to apply method of Separation of Variables to solve the differential equation. This is one of the most common method use to solve differential equation in multiple variables.

In this method we will assume that $\psi(x,t)$ can be written as a product of two functions which are only function of x or t but not both. After finding separable solutions we can write any physically allowed general solution as a sum of separable solutions (sometime we need infinitely many of them). Let,

$$\psi(x,t) = f(x) \times g(t) \tag{6}$$

be a separable solution. Show that the separable solution of the differential equation are of the form:

$$f(x) = f_1 e^{\frac{-i}{\hbar}kx} + f_2 e^{\frac{i}{\hbar}kx} \tag{7}$$

$$g(t) = g_1 e^{\frac{-i}{\hbar}lt} + g_2 e^{\frac{i}{\hbar}lt} \tag{8}$$

Find the relation between k and l in terms of α and e.

Hint:

- You might find it helpful to read `https://andrealommen.github.io/PHY309/lectures/separation`

6.2.2

Can you identify (/guess) what does the various arbitrary constants in the separable solution of above form symbolize in reality.

6.3: Particles with non-zero spin also satisfy this equation

6.3.1

It is only that in case of particles with zero spin it gives complete description of the evolution of wave function. In other cases this equation is not enough to describe the wave function but whatever the wave equation is it will also satisfy our differential equation. Thus we can say that all one dimensional wave function that exist in reality satisfy our differential equation.

Let's take Dirac equation in 1 dimension for instance.

The wave function (ψ) for a spin half particle in 1 dimension has two components, $\psi = \begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix}$

Both components satisfy the following first order differential equations:

$$i\hbar\frac{\partial\psi_1}{\partial t} = m\psi_1 - i\hbar\frac{\partial\psi_2}{\partial x} \tag{9}$$

$$i\hbar\frac{\partial\psi_2}{\partial t} = -m\psi_2 - i\hbar\frac{\partial\psi_1}{\partial x} \tag{10}$$

Show that both ψ_1 and ψ_2 satisfy our differential equation.

Also find the value of $\frac{\alpha}{e}$ by comparing our differential equation with the differential equation obtained in this question.

6.4: Deriving Schrödinger equation from our our differential eq.

6.4.1

In this section we will show that Schrödinger equation is same as our differential equation in the non-relativistic limit.

Let $\psi(x,t) = Ae^{\frac{-i}{\hbar}(lt-kx)}$

Let $\phi(x,t) = \psi(x,t) \cdot e^{\frac{i}{\hbar}mt}$

Here ϕ only differs from ψ in phase. Since probability density only depends upon the magnitude square of the wave function the probability density given by both the wave function at any given (x,t) will be the same.

Show that $\phi(x,t)$ satisfies the following differential equation in the limit k⋘m :

$$\frac{-\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} = i\hbar\frac{\partial\phi}{\partial t} \tag{11}$$

Note that the above differential equation is same as Schrödinger equation with no potential term (as expected for a non-relativistic free particle).

Hint:

- You might need to use the binomial approximation https://en.wikipedia.org/wiki/Binomial_approximation

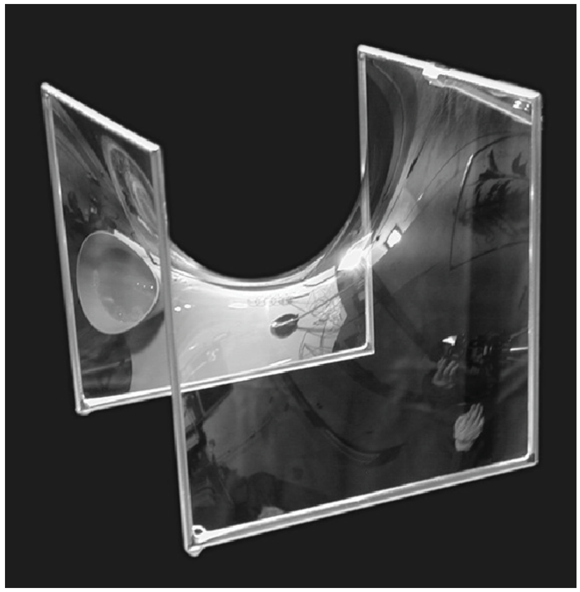


Figure 3: soap film due to wire frame

Minimal surfaces

Introduction

7. A minimal surface is a regular surface which has mean curvature zero everywhere.
(regular= having nice properties like:
1. tangent plane defined everywhere
2. Does not intersect itself)
An equivalent definition is the following: A minimal surface is a critical point for area given all the surfaces sharing the same boundary.
Because of this property, minimal surfaces (and their generalizations) find applications in materials science, general relativity, field theories, and even architecture and art.

Question

In particular, soap films not enclosing any volume try to minimize surface energy.
Thus, soap films formed due to a wire-frame resemble minimal surfaces. (deviation due to gravity and other forms of interaction energy in the soap film)

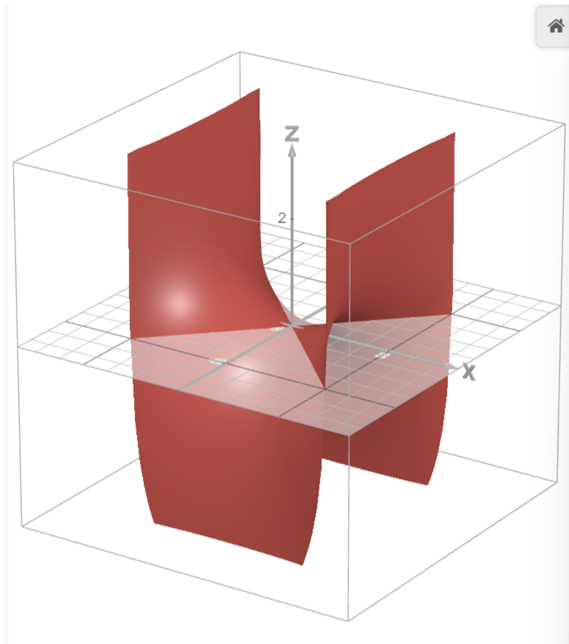


Figure 4: First Scherk Surface

Figure 3 shows the soap film formed due to a bent rectangular frame.
The surface is called the **First Scherk Surface**. The equation is given by :

$$z = a\left(ln(cos(\frac{y}{a})) - ln(cos(\frac{x}{a}))\right)$$

Prove that it is indeed a minimal surface (using any of the two definitions provided)

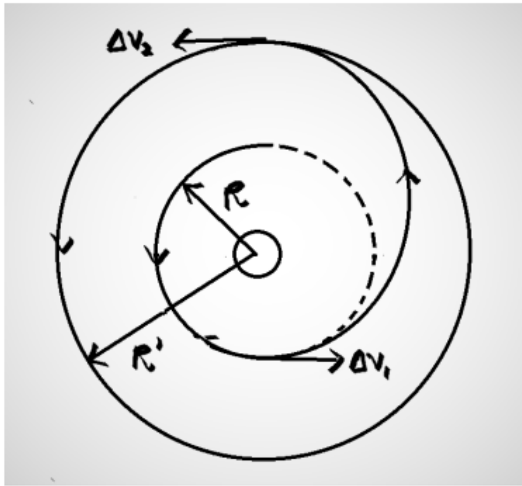


Figure 5: Hohmann Orbit Transfer

Orbital Mechanics

8. We have different orbital transfer mechanisms which are essentially orbital maneuvers, used to transfer spacecraft between two different orbits.
As depicted in Figure 5, one can visualize a satellite orbit transfer. The satellite is being transferred from one circular orbit to another using an arbitrary path.
Hohmann transfer is another orbital maneuver.
For the final figure, Figure 6, you are given a satellite orbiting the moon of a planet in a highly elliptical orbit (Use H as the height of the atmosphere and radius R as the the radius of the moon).

   (a) **Refer to figure 5**
   Assuming that the arbitrary path is elliptical, answer the following questions. Changes in velocities (provided by the boosters) at points 1 and 2 are the required velocities for the transfer of orbits; shown by Δv_1 and Δv_2 respectively

      i. Find Δv_1 and Δv_2. (Compute it under the assumption of instantaneous impulse)
      ii. What is the eccentricity of the ellipse?
      iii. What is the time taken in the transfer?
      iv. Find a general equation for velocity in terms of r (radius of a circle it's orbiting) and a (length of the semi-major axis).

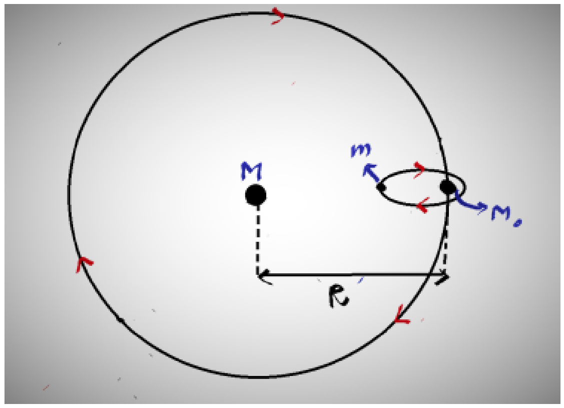


Figure 6: Moon Orbit

*Assume the mass of the planet to be M and the mass of the satellite to be m.

(b) **Refer to figure 6**

i. Find the maximum possible value of eccentricity.

ii. Find the corresponding apogee distance.
Assume that all the orbits are in the same plane.